

NEW ALGORITHMS FOR OPTIMIZING THE SIZES OF
DIXON AND DIXON DIALYTIC MATRICES

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NEW ALGORITHMS FOR OPTIMIZING THE SIZES OF
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Dedicated to my beloved family especially my wife Fatemeh, my lovely son Behnia, my parents and my parents-in-law. Thank you very much for being supportive, helpful and understanding.

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ABSTRACT

Extraneous factors are unwanted parameters in the resulting polynomial which is extracted, in the process of eliminating variables in a symbolic polynomial system. The main aim of this research is to reduce the number extraneous factors via optimising the size of Dixon matrices and its modified version, Dixon Dialytic matrices. This enhances the process of computing the *resultant* which is a tool for solving polynomial equations. An optimisation algorithm has been designed based on the sensitivity of the size of the Dixon matrix to the support set of the associated polynomial system. Some new polynomials introduced in this research are used to replace the original polynomials in the system to suppress the effects of the supports of the polynomials on each other. Moreover, in order to find the best position of the support hulls, in relation to each other, appropriate monomial multipliers were constructed and used to multiply each of the polynomials. The Dixon matrix for a generic mixed polynomial system can be optimised using these multipliers. Furthermore, the optimisation of the size of Dixon Dialytic matrix calls for the computation of an optimal arbitrary parameter g in the construction of the matrix and the support set c of monomial multipliers so that all the Dixon Dialytic sub-matrices are minimised by considering the relationship between the sizes of the corresponding Dixon matrices. Thus, the monomial multipliers that are obtained during the optimization process of the Dixon matrix are used for minimising of the Dixon Dialytic matrix as well; then the search for optimal monomial g is initiated in the intersection region of the associated convex hulls of the polynomial system. Appropriate choices of g enables further reduction in the size of the matrix. The constructed optimisation algorithms have been analysed in terms of complexity and found to be at least comparable with the existing competing methods of Chtcherba. The results of the implementation of these methods on standard examples reveal the superiority of the new methods and demonstrate no failure in optimising the size of the Dixon matrices compared to Chtcherba's.

ABSTRAK

Faktor berlebihan merupakan parameter yang tidak dikehendaki dalam polinomial yang terhasil daripada proses menghapuskan pembolehubah bagi sistem polinomial berbentuk simbolik. Matlamat utama tesis ini ialah mengurangkan faktor berlebihan ini dengan cara mengotimumkan saiz matriks Dixon dan versi matrix Dixon yang terubahsuai, matriks Dixon dialitik. Langkah ini akan meningkatkan proses pengiraan *resultant*, suatu alat untuk menyelesaikan sistem persamaan polinomial. Satu alkhwarizmi pengotimuman telah dihasilkan berdasarkan kepekaan saiz matriks Dixon terhadap set sokongan bagi sistem polinomial yang berkaitan. Beberapa polinomial baru yang diperkenalkan dalam penyelidikan ini telah digunakan untuk menggantikan polinomial asal dalam sistem bagi mengurangkan kesan set-set sokongan ke atas satu sama lain. Tambahan pula, untuk menentukan kedudukan yang terbaik bagi hul-hul sokongan, secara relatif antara satu sama lain, pendarab monomial yang sesuai telah dibina dan digunakan untuk mendarab setiap polinomial. Matriks Dixon bagi sistem polinomial bercampur dalam bentuk generik boleh dioptimumkan menggunakan pendarab-pendarab tersebut. Untuk mengoptimumkan saiz matriks Dixon dialitik pula, pengiraan perlu dilakukan untuk mengira parameter sebarangan g yang optimal untuk pembinaan matriks yang berkenaan, serta set sokongan c bagi pendarab-pendarab monomial, supaya semua submatriks Dixon dialitik dapat diminimumkan dengan mempertimbangkan hubungan antara saiz bagi matriks-matriks Dixon yang berpadanan. Oleh itu, pendarab-pendarab monomial yang dihasilkan dalam proses pemngotimuman saiz matriks Dixon juga digunakan untuk mengotimumkan saiz matriks Dixon dialitik; kemudian, penggelitaran bagi parameter optimal g dimulakan di rantau persilangan hul-hul cembung yang berkaitan bagi sistem polinomial itu. Pilihan yang sesuai bagi g memungkinkan pengurangan tambahan dalam saiz matriks. Alkhwarizmi pengotimuman yang dihasilkan dalam kerja penyelidikan ini telah dianalisis dari segi kerencaman dan didapati sekurang-kurangnya setanding dengan kaedah mutakhir yang telah dihasilkan Chtcherba. Hasil pelaksanaan alkhwarizmi-alkhwarizmi yang dihasilkan untuk contoh-contoh piawai menunjukkan keunggulan alkhwarizmi-alkhwarizmi baru ini serta tidak pernah gagal mengotimumkan saiz matriks Dixon, berbanding kaedah Chtcherba.

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LIST OF SYMBOLS

\mathcal{A}	– Support of a polynomial system
B_V	– The degree of the resultant over variety V
\mathbb{C}^d	– d -dimensional vector space over complex number
\mathbb{C}^*	– Complex numbers without origin
$D(y_1, y_2, \dots, y_n)$	– A polynomial produced by the set $\{y_1, y_2, \dots, y_n\}$
$\deg f$	– The total degree of a polynomial f
$\deg_x f$	– The degree of a polynomial f in terms of variable x
$\Delta_{\mathcal{A}}$	– Support of Dixon polynomial of generic polynomial system with support \mathcal{A}
$\det(M)$	– The determinant of square matrix M
\mathcal{F}	– Polynomial system
$\mathbb{K}[y_1, y_2, \dots, y_d]$	– Ring of polynomials with coefficients from domain \mathbb{K}
\mathbb{N}^d	– Set of d -tuples with nonnegative integer values
$\text{minor}_{\max}(M)$	– A maximal size non-singular submatrix of M
$\mu_d(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_d)$	– Mixed volume of polytopes $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_d$
$\mathcal{P} + \mathcal{Q}$	– Minkowski sum of supports
$\text{Res}_V(\mathcal{F})$	– Resultant of polynomial system \mathcal{F} over projective variety
SPARC station 10	– A workstation computer made by Sun Microsystems announced in May 1992
SUN-4 workstation	– A series of Unix workstations and servers produced by Sun Microsystems, launched in 1987

$\theta(f_0, \dots, f_d)$	– The Dixon polynomial
$\theta_i(g)$	– The Dixon polynomial of $\{f_0, \dots, f_{i-1}, g, f_{i+1}, \dots, f_d\}$
Θ	– The Dixon matrix
Vol_d	– d-dimensional Euclidean volume
x^α	– A monomial $x_1^{\alpha_1} \dots x_d^{\alpha_d}$, where $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$
\mathbb{Z}^d	– Set of d -tuple with integer values
$\mathbb{Z}[a_1, a_2, \dots]$	– Ring of polynomials with coefficient set $\{a_1, a_2, \dots\}$ from \mathbb{Z}
$\mathbb{Z}[a_1, a_2, \dots][x_1, x_2, \dots]$	– Ring of polynomials with coefficient set $\{a_1, a_2, \dots\}$ from \mathbb{Z} and variable set $\{x_1, x_2, \dots\}$
$Z_V(f_1, \dots, f_d)$	– Set of solutions polynomials f_1, \dots, f_d have in common in variety

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CHAPTER 1

INTRODUCTION

1.1 Background of study

In recent years, the search for finding efficient algorithms for solving systems of polynomials has received renewed attention due to their importance in both practical and theoretical interests including robotics, kinematics, computational number theory, solid modeling, quantifier elimination and geometric reasoning problems. One possible theory, which is commonly used to solve such a systems, is elimination of the variables. Evidence to this is recent breakthroughs in elimination theory such as development of fast algorithms for solving polynomial systems with finitely many solutions (Faug'ere *et al.*, 1992), variations of classical constructive techniques for eliminating the variables (Boole, 2003; Canny, 1990), development of elimination methods, which exploit the structure of polynomial systems to solve them efficiently (Canny and Pedersen, 1993; Canny and Emiris, 2000; Emiris, 1994; Sturmfels, 1991) and development of efficient techniques for numerically solving nonlinear systems (Morgan, 2009; Verschelde et al., 1994).

The currently known elimination techniques for solving a polynomial system are classified into symbolic, numeric and geometric techniques. Matrix methods, characteristic set construction and Gröbner basis construction are three methods which are applied in symbolic technique (Chtcherba, 2003).

Let

$$\begin{cases} f_1(x_1, \dots, x_n) = 0 \\ f_2(x_1, \dots, x_n) = 0 \\ \vdots \\ f_m(x_1, \dots, x_n) = 0 \end{cases}, \quad (1.1)$$

be a system of m polynomials in n variables with the real coefficients. Solving such a system means to determine a set of n -dimensional points (a_1, \dots, a_n) , which satisfy all of the polynomials in the system. If a system of polynomials is a symbolic or parametric form, it means that some of the coefficients or all of them are arbitrary parameters, which are chosen from specified number set. The problem of solving such a polynomial system is to find conditions on the coefficients under which the system has a set of solutions called **resultant**.

The Gröbner basis algorithm and the characteristic set method are not streamlined for finding resultant (Kapur and Saxena, 1995). They can rewrite a given polynomial system in such a way that the resultant can be easily extracted. Even though these methods compute resultants, they are not suitable in general for that purpose, because even on simple examples, these methods take an unpredictable time to compute, often running out of memory (Kapur *et al.*, 1995). Example 1.1.1 from (Chtcherba, 2003) highlights this disadvantage in case of Gröbner basis algorithm.

Example 1.1.1 (Li *et al.*, 1989) *The Gröbner basis computation of the following polynomial system, presented in*

$$\mathcal{F} = \begin{cases} f_0 = x^3y^2 + c_1x^3y + y^2 + c_2x + c_3 \\ f_1 = c_4x^4y^2 - x^2y + y + c_5 \\ f_2 = u_0 + u_1x + u_2y \end{cases},$$

where x, y are variables and $c_1, c_2, c_3, c_4, c_5, u_0, u_1$ and u_2 are parameters, quickly runs out of memory before computation finishes. This computation is done using a

system with 2GB memory. Yet the 12×12 resultant matrix can be set up in around 10 seconds using other methods by same system.

On the other hand, the effectiveness of matrix-based methods has been demonstrated in a variety of applications (Chionh, 1990; Emiris, 1994; Kapur et al., 1995; Kapur et al., 1994; Manocha, 1992; Michel et al., August 1996; Sederberg and Goldman, 1986), and the methods involving resultants offer an acceptable real-time solution (Manocha, 1992). Therefore, in case of symbolic polynomial systems, the matrix based methods are preferred.

There are two major classes of Matrix-based constructions to compute univariate or multivariate resultants called Sylvester type and Bezout-Cayley type constructions (Chtcherba and Kapur, 2003). Dixon method (Dixon, 1908), that is the subject of this thesis, is of Bezout-Cayley type, while its modified method, Dixon Dyalitic (Chtcherba, 2003), is a combination of two above major types. All of these formulations try to eliminate n variables from $n + 1$ polynomials by constructing resultant matrices.

Dixon method can be considered as one of the best matrix-based methods for finding the polynomial including the resultant of a polynomial system (Chtcherba, 2003) which is called **projection operator**. This capability is due to certain properties such as producing a dense resultant matrix implied as not existence of a great number of zeros in rows and columns of the matrix and also producing small resultant matrix considered as lower dimension of created matrix. Besides, the Dixon method's being uniform implied as computing the projection operator directly without considering a particular order of variables, and also the method's being automatic referred as eliminating all the variables at the same time (Chtcherba, 2003; Kapur et al., 1994) can be taken into consideration.

All multivariate resultant methods, except in some special cases (Faug'ere *et al.*, 1992), compute a non-negligible multiple of the resultant which does not provide any information about the solutions of the polynomial system in hand (Saxena,

1997). In particular, they just compute the multiplicative product of the resultant with some extraneous factors. These extraneous factors are undesirable and they create problems in certain applications (Chtcherba, 2003; Saxena, 1997). Dixon method has also suffered by this disadvantage (Chtcherba and Kapur, 2004b; Chtcherba and Kapur, 2003).

A useful property of Dixon method is the sensitivity of Dixon matrix size to exponents of the variables in the polynomials of a polynomial system. In other words, if we change the power of the variables in the polynomials of a system, the Dixon matrix size changes. Another remarkable property of Dixon formulation is that, the computational complexity of Dixon formulation is not governed by the total degree of polynomials, unlike the Macaulay resultants (Kapur and Saxena, 1996). See Example 1.1.2.

Example 1.1.2 *In the following polynomial system,*

$$\mathcal{F} = \begin{cases} f_0 = a_1 + a_2x + a_3y \\ f_1 = b_1y^2 + b_2x^2 + b_3x^3y, \\ f_2 = c_1 + c_2xy^2 + c_3x^2y \end{cases}$$

the size of Dixon matrix is 8×9 and Dixon polynomial has 50 terms (monomials). By multiplying x^3y^2 to f_0 , x^2y to f_1 and x to f_2 , we have a converted system of polynomials as

$$\mathcal{F}' = \begin{cases} x^3y^2 f_0 = a_1x^3y^2 + a_2x^4y^2 + a_3x^3y^3 \\ x^2y f_1 = b_1x^2y^3 + b_2x^4y + b_3x^5y^2, \\ x f_2 = c_1x + c_2x^2y^2 + c_3x^3y \end{cases},$$

and the Dixon matrix size increase to 14×15 with Dixon polynomial, which includes 72 terms. While by choosing multipliers x^3y^2 , x^2y and x^3y for f_0 , f_1 and f_2 , respectively, the size of Dixon matrix decreases again to 9×9 with 39 monomials in Dixon polynomial.

As illustrated in Example 1.1.2 above, the Dixon matrix responds to any conversion of the exponents of the polynomials in the polynomial system. As long as the Dixon matrix has this property, it is possible to optimize (minimize) the size of the Dixon matrix. As future illustrations, the size of Dixon matrix has direct dependency on the number of extraneous factors, which implies that the fewer dimension of the Dixon matrix results in fewer extraneous factors and vice versa.

Using this property, this thesis is intended to optimize the size of Dixon matrix in order to decrease the solving process and getting more accurate in process of finding results.

1.2 Preliminary concept

This section introduces some basic concepts, which are needed to develop a resultant and its formulation.

Consider a multivariate polynomial in the expanded form

$$f(x_1, x_2, \dots, x_d) = \sum_{\alpha=(\alpha_1, \dots, \alpha_d) \in \mathcal{A}} c_{\alpha} x_1^{\alpha_1} \dots x_d^{\alpha_d}, \text{ where } c_{\alpha} \neq 0, \quad (1.2)$$

with $c_{\alpha} x_1^{\alpha_1} \dots x_d^{\alpha_d}$ as a monomial, which can be written in the simplified form as

$$f(x_1, x_2, \dots, x_d) = \sum_{\alpha \in \mathcal{A}} c_{\alpha} \mathbf{x}^{\alpha}, \text{ where } c_{\alpha} \neq 0,$$

The set $\mathcal{A} \subset \mathbb{N}^d$, which is a finite set of exponents, is called the support of the polynomial. The coefficients c_{α} for $\alpha \in \mathcal{A}$ are polynomials over \mathbb{Z} in the

parameters so that we can write $c_\alpha \in \mathbb{Z}[\mathbf{c}]$, where \mathbf{c} is the vector of parameters. Hence we write $f \in \mathbb{Z}[\mathbf{c}, \mathbf{x}]$ or $f \in \mathbb{Z}[\mathbf{c}][\mathbf{x}]$.

The total degree of a multivariate polynomial f is the maximum degree of any monomial in f , where the degree of a particular monomial is the sum of the variable exponents.

Example 1.2.1 *The polynomial*

$$f(x, y) = (a + 2)x^2 - (b - c - 1)xy + c^2y^2 - abx - 1,$$

has support $\mathcal{A} = \{(2,0), (1,1), (0,2), (1,0), (0,0)\}$ with coefficients over $\mathbb{Z}[a, b, c]$, hence $f \in \mathbb{Z}[a, b, c][x, y]$. The terms $(a + 2)x^2$, $(b - c - 1)xy$, c^2y^2 , abx and 1 are called monomials of the polynomial and the total degree of f is $\max\{2, 1, 0\} = 2$.

To link algebra and geometry, we will study polynomials over a field. The basic intuition is that a field is a set where one can define addition, subtraction, multiplication, and division with the usual properties. Standard examples are the real numbers \mathbb{R} and the complex numbers \mathbb{C} , whereas the set integers \mathbb{Z} is not a field since division fails (3 and 2 are integers, but their quotient $3/2$ is not). A formal definition of field can be found in (Baez et al., 1992; Cox et al., 2006).

We now introduce affine space and projective space.

Definition 1.2.1 (Chtcherba, 2003) *Consider a field \mathcal{K} and a positive integer n , define the n -dimensional affine space over \mathcal{K} to be the set*

$$\mathcal{K}^n = \{(a_1, \dots, a_n) | a_1, \dots, a_n \in \mathcal{K}\}.$$

For an example of an affine space, consider the case $\mathcal{K} = \mathbb{R}$. A coordinate system for the n -dimensional affine space \mathbb{R}^n is determined by any basis of n vectors, which are not necessarily orthogonal. Therefore, the resulting axes are not necessarily mutually perpendicular or have the same unit measure. In this sense, affine space is a generalization of Cartesian or Euclidean space. In general, we call $\mathcal{K}^1 = \mathcal{K}$ the affine line and \mathcal{K}^2 the affine plane.

Definition 1.2.2 (Chtcherba, 2003) *Consider a field \mathcal{K} , and f_1, \dots, f_n be polynomials in $\mathcal{K}[x_1, \dots, x_d]$.*

$$V(f_1, \dots, f_n) = \{(a_1, \dots, a_d) \in \mathcal{K}^d \mid \forall i = 1, \dots, n, f_i(a_1, \dots, a_d) = 0\},$$

*is called the **affine variety** of f_1, \dots, f_n .*

An affine variety is the set of all solutions of the system of polynomials in an affine space. We will use the letters V, W , etc, to denote affine varieties. In case of $\mathcal{K} = \mathbb{R}$, the conic sections studied in analytic geometry (circles, ellipses, parabolas, and hyperbolas) are affine varieties. Likewise, graphs of polynomial functions are affine varieties (the graph of $y = f(x)$ is $V(y - f(x))$). More examples of affine variety can be found in (Cox *et al.*, 2006).

Definition 1.2.3 (Saxena, 1997) *The **projective plane** over \mathbb{R} , denoted $\mathbb{P}^2(\mathbb{R})$, is the set*

$$\mathbb{P}^2(\mathbb{R}) = \mathbb{R}^2 \cup \{\text{one point at } \infty \text{ for each equivalence class of parallel lines}\}$$

where an equivalence class L/\sim of a line L consists of all lines parallel to the line L .

We let $[L]_\infty$ denote the common point at ∞ of all lines parallel to L . Then we call the set $\bar{L} = L \cup [L]_\infty \subset \mathbb{P}^2(\mathbb{R})$ the projective line corresponding to L . Note that two projective lines always meet at exactly one point: if they are not parallel, they meet at a point in \mathbb{R}^2 ; if they are parallel, they meet at their common point at ∞ .

Thus far, our discussion of the projective plane has introduced some nice ideas, but it is not entirely satisfactory. For example, it is not really clear why the line at ∞ should be called a projective line. A more serious objection is that we have no unified way of naming points in $\mathbb{P}^2(\mathbb{R})$. Points in \mathbb{R}^2 are specified by coordinates, but points at ∞ are specified by lines. To avoid this asymmetry, we will introduce **homogeneous coordinates** on $\mathbb{P}^2(\mathbb{R})$.

To get homogeneous coordinates, we will need a new definition of projective space. The first step is to define an equivalence relation on nonzero points of \mathbb{R}^3 by setting

$$(x_1, y_1, z_1) \sim (x_2, y_2, z_2),$$

if there is a nonzero real number λ such that $(x_1, y_1, z_1) = \lambda(x_2, y_2, z_2)$. One can easily check that \sim is an equivalence relation on $\mathbb{R}^3 - \{\mathbf{0}\}$ (where $\mathbf{0}$, refers to the origin $(0,0,0)$ in \mathbb{R}^3). Then we can redefine projective plane as follows.

Definition 1.2.4 (Chtcherba, 2003) $\mathbb{P}^2(\mathbb{R})$ is the set of equivalence classes of \sim on $\mathbb{R}^3 - \{\mathbf{0}\}$. Thus, we can write

$$\mathbb{P}^2(\mathbb{R}) = (\mathbb{R}^3 - \{\mathbf{0}\})/\sim,$$

If a triple $(x, y, z) \in \mathbb{R}^3 - \{\mathbf{0}\}$ corresponds to a point $p \in \mathbb{P}^2(\mathbb{R})$, we say that (x, y, z) is a **homogeneous coordinate** of p .

A clarification on the equivalence of above two definitions of projective plane is presented in (Cox *et al.*, 2006).

Homogeneous coordinates are different from the affine coordinates in that the former are not unique. Indeed the distinction between affine and projective spaces arises especially when comparing coordinates. For example, the triples $(1, -2, 3)$ and $(-2, 4, -6)$ are the affine coordinates of two distinct points of the affine space \mathbb{R}^3 , but are the projective coordinates of the same point of the projective plane $\mathbb{P}^2(\mathbb{R})$, since projective coordinates are determined up to proportionality.

The construction of the real projective plane given above can be generalized to yield projective spaces of any dimension n over any field \mathcal{K} . We define an equivalence relation \sim on the nonzero points of \mathcal{K}^{n+1} by setting

$$(x'_0, \dots, x'_n) \sim (x_0, \dots, x_n),$$

if there is a nonzero element λ such that $(x'_0, \dots, x'_n) = \lambda(x_0, \dots, x_n)$. If we let $\mathbf{0}$ denote the origin $(0, \dots, 0)$ in \mathcal{K}^{n+1} , then the projective space is defined as follows.

Definition 1.2.5 (Kapur *et al.*, 1994) An n -dimensional **projective space** over the field \mathcal{K} , denoted $\mathbb{P}^n(\mathcal{K})$, is the set of equivalence classes of \sim on $\mathcal{K}^{n+1} - \{\mathbf{0}\}$. Thus,

$$\mathbb{P}^n(\mathcal{K}) = (\mathcal{K}^{n+1} - \{\mathbf{0}\})/\sim.$$

Each nonzero $(n + 1)$ -tuple $(x_0, \dots, x_n) \in \mathcal{K}^{n+1}$ defines a point p in $\mathbb{P}^n(\mathcal{K})$, and we say that (x_0, \dots, x_n) are homogeneous coordinates of p .

Like $\mathbb{P}^2(\mathbb{R})$, each point $p \in \mathbb{P}^n(\mathcal{K})$ has many sets of homogeneous coordinates. For example, in $\mathbb{P}^3(\mathbb{C})$, the homogeneous coordinates $(0, \sqrt{2}, 0, i)$ and $(0, 2i, 0, -\sqrt{2})$ describe the same point since $(0, \sqrt{2}, 0, i) = \sqrt{2}i(0, 2i, 0, -\sqrt{2})$. In general, we will write $p = (x_0, \dots, x_n)$ to denote that (x_0, \dots, x_n) are homogeneous coordinates of $p \in \mathbb{P}^n(\mathcal{K})$. We can think of $\mathbb{P}^n(\mathcal{K})$ geometrically as the set of lines through the origin in \mathcal{K}^{n+1} .

Definition 1.2.6 (Chtcherba, 2003) *A polynomial $f \in \mathcal{K}[x_1, \dots, x_d]$ of a certain total degree n is **homogeneous** if every monomial term in f is of degree n .*

Definition 1.2.7 (Chtcherba, 2003) *Let \mathcal{K} be a field, and let f_1, \dots, f_n be homogeneous polynomials in $[x_1, \dots, x_d]$. Then*

$$V(f_1, \dots, f_n) = \{(a_1, \dots, a_d) \in \mathbb{P}^d(\mathcal{K}) \mid \forall i = 1, \dots, n, f_i(a_1, \dots, a_d) = 0\}$$

*is called the **projective variety** defined by f_1, \dots, f_n .*

For example, in $\mathbb{P}^d(\mathcal{K})$, any nonzero homogeneous polynomial of degree 1,

$$f(x_0, \dots, x_d) = c_0x_0 + \dots + c_dx_d,$$

defines a projective variety $V(f)$ called a hyperplane.

Now consider a system of $d + 1$ polynomials in d variables, which we shall call a "over-constrained", following the nomenclature of Chtcherba (Chtcherba, 2003). (Clearly, d polynomials in d variables, as a well-constrained polynomial system, always have a solution, so the corresponding parameters are unconstrained. Therefore, $d + 1$ polynomials are considered. No generality is achieved if more polynomials are considered, in which case only a subset of $d + 1$ polynomials at a time, or a linear combination of the polynomials, has to be considered.)

The polynomial system can be written as

$$\mathcal{F} = \{f_0, f_1, \dots, f_d\} \subset \mathbb{Z}[\mathbf{c}, \mathbf{x}] , \quad (1.3)$$

where

$$f_0 = \sum_{\alpha \in \mathcal{A}_0} c_0 \mathbf{x}^\alpha \quad , \quad f_1 = \sum_{\alpha \in \mathcal{A}_1} c_1 \mathbf{x}^\alpha \quad , \dots , \quad f_d = \sum_{\alpha \in \mathcal{A}_d} c_d \mathbf{x}^\alpha . \quad (1.4)$$

The first polynomial is indexed at 0, so that the last index is equal to the number of variables, i.e. the dimension of the problem.

Let \mathcal{A}_i be the support of the polynomial $f_i \in \mathcal{F}$. We consider the collection $\mathcal{A} = \langle \mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_d \rangle$ to be the support of a polynomial system $\mathcal{F} = \{f_0, f_1, \dots, f_d\}$. Generally, we will classify polynomial systems according to the properties of their supports.

Definition 1.2.8 (Kapur *et al.*, 1995) A polynomial system $\mathcal{F} = \{f_0, f_1, \dots, f_d\}$ with support $\mathcal{A} = \langle \mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_d \rangle$ is called **unmixed** if $\mathcal{A}_0 = \mathcal{A}_1 = \dots = \mathcal{A}_d$, and **mixed** otherwise.

Definition 1.2.9 (Saxena, 1997) A set of polynomials $\{f_1, f_2, \dots, f_n\}$ are **algebraically independent** if and only if there does not exist a non-zero polynomial $D(y_1, y_2, \dots, y_n) \in \mathbb{K}[y_1, y_2, \dots, y_n]$ such that $D(f_1, f_2, \dots, f_n) = 0$.

In this entire thesis, we consider the polynomials in a generic form, by which we mean all the coefficients of the polynomials are algebraically independent. For example consider the system of polynomials

$$\begin{cases} f_0 = x^2 + axy - y + b \\ f_1 = xy + (a+b)y - c, \\ f_2 = bx + y + 2ac \end{cases}$$

where x, y are variables and a, b, c are parameters. This system is not generic because the coefficient of x^2 in f_0 is not independent, and also the coefficient of y in f_1 depends on coefficients in f_0 .

Definition 1.2.10 (Emiris, 1994) *The objective of eliminating the variables for solving a polynomial system \mathcal{F} is obtaining a polynomial purely in the coefficients of \mathcal{F} . The vanishing of such polynomials is a necessary condition for the existence of solutions for \mathcal{F} . We will call any such polynomial a **projection operator**.*

One of the simplest cases of a polynomial system \mathcal{F} is when all polynomials are of total degree 1, which corresponds to following linear system

$$\mathcal{F} = \begin{cases} f_0 = c_{0,1}x_1 + c_{0,2}x_2 + \cdots + c_{0,d}x_d + c_{0,d+1} \\ f_1 = c_{1,1}x_1 + c_{1,2}x_2 + \cdots + c_{1,d}x_d + c_{1,d+1} \\ \vdots \\ f_d = c_{d,1}x_1 + c_{d,2}x_2 + \cdots + c_{d,d}x_d + c_{d,d+1} \end{cases}, \quad (1.5)$$

where each $c_{i,j}$ is a real coefficient or parameter. Note that if the set of variables be considered as $\{x_1, x_2, \dots, x_d, 1\}$, we can rewrite the above system in the matrix form with coefficient matrix of size $d \times d$ as follows.

$$\mathcal{F} = \begin{pmatrix} c_{0,1} & c_{0,2} & \cdots & c_{0,d+1} \\ c_{1,1} & c_{1,2} & \cdots & c_{1,d+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{d,1} & c_{d,2} & \cdots & c_{d,d+1} \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ 1 \end{pmatrix} = 0.$$

From linear algebra, we know that the solution of \mathcal{F} exists in affine space \mathbb{C}^d if and only if the following determinant vanishes

$$R = \begin{vmatrix} c_{0,1} & c_{0,2} & \cdots & c_{0,d+1} \\ c_{1,1} & c_{1,2} & \cdots & c_{1,d+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{d,1} & c_{d,2} & \cdots & c_{d,d+1} \end{vmatrix}.$$

The determinant R will be called the projection operator and as we shall see later, is the resultant. The vanishing of R provides a necessary and sufficient condition for the above linear polynomial system to have a nontrivial solution.

A resultant for a polynomial system may be reducible under certain conditions relating to its coefficients. A simple example is the resultant of $ax + b = 0$ and $bx + a = 0$ which with respect to x is $(a - b)(a + b)$. A point arises then as to the irreducibility of the resultant for a generic polynomial system.

The presence of extraneous factors is closely related to the projection operator. As matter of fact the extraneous factors are all of the factors in a projection operator apart from the resultant. In other words, in a projection operator of a polynomial system, the extraneous factors are all of the factors in the ratio of the projection operator and the resultant. Example 1.2.2, which follows, illustrates the case.

Example 1.2.2 *Consider the strophoid, a curve widely studied by mathematicians of 17th and 18th century, which can be written in a parametric form as follows.*

$$x = a \sin(t), \quad y = a \tan(t)(1 + \sin(t)),$$

where a is a constant. The graph is shown in Figure 1.1.

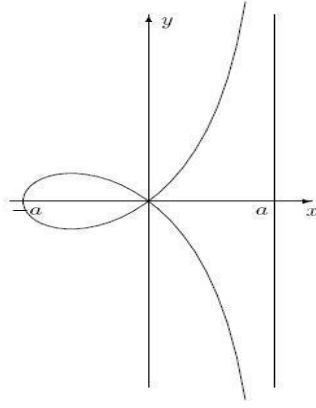


Figure 1.1 Strophoid for Ex 1.2.2.

To find an implicit equation for the strophoid using resultant, we have to restate the equations in terms of polynomials instead of trigonometric functions, as follows.

Letting $S = \sin t$, $C = \cos t$, $T = \tan t$, the trigonometric equations of the strophoid can be written as

$$\begin{cases} f_0 = C^2 + S^2 - 1 = 0 \\ f_1 = CT - S = 0 \\ f_2 = x - aS = 0 \\ f_3 = y - aT(1 + S) = 0 \end{cases} . \quad (1.6)$$

We wish to eliminate C , S and T to get a single equation in terms of x, y and a . We can treat the monomials C^2, S^2, CT, S, T and ST as separate variables to make the above equations linear so that we can employ the tools of Linear Algebra. Unfortunately, in this case, the number of variables is six and the number of equations is only four that is the system is not well-constrained. To get more equations, we can pre-multiply some of the polynomials by some monomials; for example, $Tf_0 = 0$ will still have all solutions of $f_0 = 0$; hence we can add it to the set of equations (Such method of adding more equations to polynomial systems is usually referred as the Dialytic method which was first used by James Joseph Sylvester in (Auzinger and Stetter, 1988; Sylvester, 1853)). As it can be seen in the

next chapter, there is lower bound on the size of the presented resultant matrix in any Dialytic resultant formulation which for above polynomial system it is 9 (computed in (Chtcherba, 2003) in page 220). It means that for this example, any Dialytic formulation does not admit resultant matrix smaller than 9×9 . To be sure, for some of the most important ones, presented in the ensuing chapters, such as Dixon Dialytic method, Macaulay method, Subdivision method and Incremental method, the size of resultant matrix for this particular example is 10×10 , 35×31 , 11×11 and 17×14 , respectively. As a sample, if we multiply f_0 by T , f_1 by C and 1, f_2 by CT , ST , C, T and 1, and f_3 by C and 1 (which can be obtained using Dixon Dialytic method presented in chapter 5) we get the following system of 10 polynomials

$$\mathcal{F}' = \begin{cases} Tf_0 = C^2T + S^2T - T \\ Cf_1 = C^2T - CS \\ f_1 = CT - S \\ CTf_2 = xCT - aCST \\ STf_2 = xST - aS^2T \\ Cf_2 = xC - aCS \\ Tf_2 = xT - aST \\ f_2 = x - aS \\ Cf_3 = yC - aCT - aCST \\ f_3 = y - aT - aST \end{cases},$$

which contains exactly 10 monomials. If we treat each monomial as independent variable, we obtain the following 10×10 linear system (in matrix form):

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -a & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 & x \\ -a & 0 & 0 & -a & y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 & -a & y \end{bmatrix}}_M \times \underbrace{\begin{bmatrix} CST \\ C^2T \\ CS \\ CT \\ C \\ S^2T \\ ST \\ S \\ T \\ 1 \end{bmatrix}}_X = 0.$$

If we consider the above linear system as $\mathcal{F}' = M \times X$, we have 10 linear equations in 10 variables. Linear algebra states that a non-trivial solution exists for \mathcal{F}' if and only if the determinant of the matrix M is zero. Since the solutions of the original polynomial system \mathcal{F} are also solutions of \mathcal{F}' , it follows that the vanishing of the determinant of M can present a condition for the original polynomial system \mathcal{F} to have a solution. In general, not every solution of \mathcal{F}' is a solution of the original polynomial system, hence the condition of vanishing of the determinant is only necessary but not sufficient for the original polynomial system to have a solution. The factors, which account for these extra solutions are called **extraneous factors**. Hence, the determinant of the above matrix must be a multiple of the resultant. The determinant of the coefficient matrix, denoted by matrix M , for the above linear system, is

$$\det(M) = -a^3(a+x)(x^3+ax^2+xy^2-ay^2).$$

Note that although the 10×10 linear system has a solution whenever $a = 0$ or $a = -x$ or when $y^2(a-x) = x^2(a+x)$, the original polynomial system will only have a solution when $y^2(a-x) = x^2(a+x)$. The solutions $a = 0$ and $a = -x$, are already accounted for in $y^2(a-x) = x^2(a+x)$. Then the factors $a = 0$ and $a = -x$ are extraneous factors. Thus equations

$$x = a \sin(t), \quad y = a \tan(t)(1 + \sin(t)).$$

are solvable for t , whenever $y^2(a-x) = x^2(a+x)$, and therefore it is an implicit equation of the strophoid.

As illustrated in Example 1.2.2 above, there are three steps in computing the resultant through a resultant matrix:

- Construction of a resultant matrix.
- Computation of the symbolic determinant.

- Identification of the resultant in the projection operator.

These three steps are common for all matrix-based method for finding resultant.

1.3 Problem formulation

Dixon (Auzinger and Stetter, 1988) defined his method for eliminating a large number of variables for a vast class of polynomial systems and computing a polynomial which includes all solutions of the polynomial system. A modified formulation of Dixon method is presented by Chtcherba (Chtcherba, 2003) which is called Dixon Dialytic. Some research found, Dixon methods show the superiority of this method in performance on a wide variety of problems(Chtcherba, 2003).

As shown in previous section since the presented resultant contains all information about the solutions of a polynomial system, the extraneous factors in a projection operator do not offer any additional information. Moreover, they make it harder for the resultant in a projection operator to be identified. Each factor in a projection operator must be checked as to whether it is an extraneous factor or is part of the resultant. This checking can be time-consuming. This issue is encountered for both Dixon and Dixon Dialytic (Chtcherba and Kapur, 2002).

The projection operator of the Dixon matrix related to Example 1.2.2 is as the following

$$\det(\max_{\text{minor}} \Theta) = a^6(a + x)(x^3 + ax^2 + xy^2 - ay^2),$$

where Θ is the Dixon matrix and $\max_{minor} \Theta$ is a sub-matrix of Dixon matrix. Note that the resultant is a factor in the determinant of a maximal minor of Dixon matrix that contains $a^6(a + x)$ as extraneous factor (see Example 1.2.2 and Example 4.3.1).

Therefore, developing or constructing techniques that can eliminate, or at least reduce, the number of extraneous factors is an important problem of research, which when solved adequately will yield handsome dividends.

1.4 Objectives of study

From the formulated problem, which is addressed in the past section, the objective of this research work could be stated as follows:

- 1) To determine how the information about the sensitivity of the size of Dixon/ Dixon Dialytic matrix with respect to the variables' power in the polynomials of the original system can be used to produce a smaller resultant matrix.
- 2) To propose a method for managing these variables' power so as to produce smaller Dixon matrix and to construct an algorithm to implement the method.
- 3) To propose an algorithm based on managing the power of the parameters existing in the Dixon Dialytic structure so as to further reduce the size of the resulting matrix.

1.5 Scope of study

The focus of this thesis is on the application of algebraic geometry, which is related to the study of geometric objects defined by polynomial equations using the tools of algebra. Although one can choose coefficients of polynomial systems from any arbitrary field of numbers, in this work, real coefficients has been considered. In order to achieve the objectives of thesis the following scope of work has been carried out:

- Identifying the applications of polynomial systems and the needs for solving them.
- Using Bezout-Cayley type construction which is taken from matrix-based method, on generic-mixed polynomial systems.
- Studying the Dixon and Dixon Diallytic formulation and their advantages and properties.
- Considering the determinant of the produced Dixon/Dixon Diallytic matrix and finding a polynomial that includes the resultant (projection operator) and its properties.
- Identifying a method to manage the total degree of the polynomials in the system in order to reduce the extraneous factors in the projection operator, which exist besides the resultant.
- Producing algorithms to eliminate or at least decrease the extraneous factors from the projection operators, implementation and analysis of the results.

1.6 Significance of study

None of the matrix-based elimination methods, in particular the Dixon formulation, produces the exact resultant of arbitrary non-generic, non-homogeneous polynomial systems. Instead, these elimination methods compute various

polynomials known as projection operators, which may contain extraneous factors besides the resultant. Since the information about the solutions of a polynomial system is completely characterized by its resultant, the extraneous factors have no additional information in a projection operator. The Dixon formulation, being the major subject of this thesis, is no exception from rule of extraneous factors. The contribution of this study is minimizing the number of extraneous factors for the Dixon based resultant formulation.

1.7 Related works

For generic polynomial systems, Macaulay devised a technique which, he determined a sub-matrix of the resultant matrix and computed its determinant to identify the extraneous factor exactly, but his method does not work in the non-generic case and for other formulations (Kapur and Saxena, 1996).

Chtcherba and Kapur presented a hypothesis for optimizing the size of Dixon matrix that is based on Corner Cutting method (Chtcherba and Kapur, 2002). They devised a heuristic for presenting the best variable order for the construction of the Dixon resultant matrix, but it stays in heuristic realm due to some limitations. Besides, in the unlikely case which is explained in their paper, the method might not present the best variable order (Chtcherba and Kapur, 2004a).

Saxena in (Kapur and Saxena, 1996) suggested a general method for reducing total degree of polynomials in the given polynomial system. His suggestion is computing the projection operator of a polynomial system with polynomials of less total degree than original polynomial system. Given a polynomial system \mathcal{F} , first one should compute c_i , the greatest common divisor (GCD) of all powers of variables x_i occurring in the non-zero terms of \mathcal{F} . Then, for all $1 \leq i \leq n$, divide each monomial in \mathcal{F} which contains $x_i^{k c_i}$, for some $k > 0$, by $x_i^{k(c_i-1)}$. Finally, compute the projection operator of the resulting smaller system. This technique

basically finds the smallest system that \mathcal{F} can scale down to, and then, works with that smaller system. Since such a procedure reduces the total degree of the input polynomials, it requires less computational resources than directly computing the projection operator of the larger system. But, this technique works when the GCD of powers are not equal to one.

Chtcherba in (Chtcherba and Kapur, 2004a) presented another heuristic for minimizing the size of Dixon matrix. He introduced some polynomial multipliers for the polynomials in a system to get smaller Dixon matrix, but his method does not work properly for general polynomial systems. Besides, in the course of implementing his method, only the number of columns of the resultant matrix is considered as a character to get a smaller Dixon matrix size. In fact, a method, which is based solely on the number of columns as the only condition contributing to the recognition of smaller matrix, is likely to be misleading.

1.8 Thesis overview

The central point of the present thesis is the notion of the resultant of a polynomial system. This thesis has been built on the success of optimizing the Dixon resultant formulation by recent researchers such as Chtcherba and Saxena. This research presents a new method for finding smaller Dixon matrix to get smaller total degree of the projection operator, which signifies less extraneous factors in the decomposed form.

In Chapter 2, the formal definitions of some matrix-based resultant formulations needed for developing the basis of this thesis are given in the univariate and multivariate cases. Dixon Dialytic formulation stems from Sylvester and Bezout-Cayley which are the most prominent formulations presented in this chapter, while Dixon method is derived from Bezout-Cayley construction.

Research methodology is presented in Chapter 3 where, the formulation procedure for algorithms of optimization of Dixon matrix and Dixon Dyalitic matrix is illustrated via flowcharts. The information about research assumption, research framework, procedures, and simulation tools are covered in this chapter. Moreover, a list of problems which are applied to test the presented algorithms and compare the achieved results of optimization methods for Dixon and Dixon Dyalitic matrices is presented.

The major objective of this research is built on the definition of Dixon formulation. Therefore, chapter 4 is assigned to illustrate the properties of this formulation by definitions and theorems. Besides, fundamental issues are raised which include the relationships between degrees of projection operator and size of Dixon matrix and effects of supports conversion on the size of Dixon matrix.

Chapter 5 is devoted to explaining the modified method of the Dixon formulation called Dixon Dyalitic. Useful details which are used for optimizing Dixon Dyalitic formulation are presented in this chapter, which include the relationships between degrees of projection operator and size of Dixon Dyalitic matrix and relation between the size of Dixon matrix and the size of Dixon Dyalitic matrix.

Chapters 6 and 7 present the main algorithms of the present research for optimizing the size of Dixon matrix and Dixon Dyalitic matrix. Furthermore, some examples are presented in order to check if the algorithms work properly. Each chapter ends with a complexity analysis of the optimization algorithms.

Chapter 8 presents some examples solved by both optimizing methods and existing methods, with which the merits of the new methods in this thesis are established. In this chapter, empirical evidence of the applicability of the proposed optimization algorithms as well as comparison with the existing algorithms is presented. The results implemented in the tables have made easier the comparison of the achieved results from the new methods and the existing methods and bounds.

Finally, summary and conclusions of the thesis, some future directions and open problems are presented in Chapter 9.

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